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A Runge–Kutta–Nyström method for the numerical integration of special second-order periodic initial-value problems

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Abstract

A new Runge–Kutta–Nyström method is developed to integrate second-order differential equations of the form $u''(t)=f(t, u)$ when they posses an oscillatory solution. Through an appropriate definition of the parameters of the method, a fourth algebraic order is obtained and the phase-lag is reduced significantly.

Keywords: Runge–Kutta–Nyström; Phase-lag; Oscillating solutions

1. Introduction

In this paper we study a special Runge–Kutta–Nyström (RKN) method of Fehlberg for the integration of systems of ODEs of the form

$$\frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k}=f(t, u(t)), \quad k=1, 2, \tag{1.1}$$

for which it is known in advance that their solution is oscillating.

The study of such systems begins with the test equation

$$\frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k}=(\mathrm{i}v)^k u(t)+c \exp(\mathrm{i}v_f t), \quad k=1, 2, \tag{1.2}$$

where c, v, v_f are real constants. In case $k=1$, van der Houwen and Sommeijer [14,16] proposed second-order m-stage methods with $m=4$, 5, 6 and phase-lag order $q=6$, 8, 10, respectively. They also derived some third-order methods with phase-lag order 6, 8, 10.

The disadvantage of these methods is the slow convergence, and their inefficiency especially in the case of forced oscillating solutions.

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For $k=1$, Sideridis and Simos [9,10] constructed known Runge–Kutta methods with minimal *phase-lag*.

For the case $k=2$ there is an extensive literature [1–5,8,11–13,15]. However, it is worth elaborating a bit more in the schemes used and presenting very accurate results as those of the present paper.

When a numerical method is applied to the test equation (1.2), a numerical approximation u_n of the exact solution $u(t_n)$ at $t_n = nh$ is obtained. This approximation is of the form

$$u_n = \sum_{j=1}^{s} k_j\left[\rho_j(vh)\right]^n + ch^k A_{\mathrm{n}}(vh, v_f h)\exp(\mathrm{i}v_f nh), \tag{1.3}$$

where s is the number of distinct roots of the characteristic polynomial of the method, ρ_j are the s distinct roots of the characteristic polynomial of the method, and k_j are the constants determined by the initial conditions.

The functions ρ_j^n and $ch^k A_{\mathrm{n}}(vh, v_f h)\exp(\mathrm{i}v_f nh)$ are called the *homogeneous* and *inhomogeneous* components of the *numerical* solution (1.3), respectively.

On the other hand, the exact solution of (1.2) is given by

$$u(t) = \sigma_1 \exp(\mathrm{i}vt) + \sigma_2 \exp(-\mathrm{i}vt) + \frac{c\exp(\mathrm{i}v_f t)}{(\mathrm{i}v_f)^k - (\mathrm{i}v)^k}, \tag{1.4}$$

where σ_1 and σ_2 are constants with $\sigma_2 = 0$ when $k=1$.

The functions $\exp(\pm\mathrm{i}vt)$ and $ch^k A_{\mathrm{e}}\exp(\mathrm{i}v_f t)$ are called the *homogeneous* and *inhomogeneous* components of the *exact* solution, respectively, where $A_{\mathrm{e}} = 1/[(\mathrm{i}v_f h)^k - (\mathrm{i}vh)^k]$.

In the phase analysis of the *homogeneous* components of (1.3) and (1.4), we have to compare the arguments (phases) of $\exp(\pm\mathrm{i}vh)$ with the arguments of the principal characteristic roots in the set $\{\rho_1, \rho_2, \ldots, \rho_s\}$. These phase errors are *time-dependent* and therefore accumulate as n increases.

In the phase analysis of the *inhomogeneous* components of (1.3) and (1.4), we have to compare the phases of A_{n} and A_{e}. These phase errors are constant in time. For this reason our study will be confined in minimizing the phase errors of the homogeneous components.

The purpose of this paper is to construct a four-stage RKN method of Fehlberg of fourth algebraic order with phase-lag order eight, in the case of $k=2$. Since our study is confined in the homogeneous phase errors, we will use as test equation

$$\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} = (\mathrm{i}v)^2 u(t). \tag{1.5}$$

2. Phase-lag analysis of RKN methods and periodicity interval

The general m-stage method for the equation

$$\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} = f(t, u(t)) \tag{2.1}$$

is of the form

$$u_n^{(0)} = u_{n-1}, \qquad u_n^{(i)} = u_{n-1} + \alpha_i h \dot{u}_{n-1} + h^2 \sum_{j=0}^{i-1} \gamma_{i,j} f\left(t_{n-1} + \alpha_j h,\, u_n^{(j)}\right), \tag{2.2}$$

$$u_n = u_n^{(m)}, \qquad \dot{u}_n = \dot{u}_{n-1} + h \sum_{j=0}^{m-1} \dot{c}_j f\left(t_{n-1} + \alpha_j h,\, u_n^{(j)}\right),$$

where $\alpha_0 = 0$ and $\alpha_m = 1$.

By applying the general method (2.2) to the test equation (1.5) we obtain the numerical solution

$$\begin{bmatrix} u_n \\ h\dot{u}_n \end{bmatrix} = D^n \begin{bmatrix} u_0 \\ h\dot{u}_0 \end{bmatrix}, \quad D = \begin{bmatrix} A(z^2) & B(z^2) \\ A'(z^2) & B'(z^2) \end{bmatrix}, \quad z = vh, \tag{2.3}$$

where A, A', B, B' are polynomials in z^2, completely determined by the parameters of the method (2.2).

The exact solution of (1.5) is given by

$$u(t_n) = \sigma_1[\exp(\mathrm{i}v)]^n + \sigma_2[\exp(-\mathrm{i}v)]^n, \tag{2.4}$$

where $\sigma_{1,2} = \frac{1}{2}[u_0 \pm (\mathrm{i}\dot{u}_0)/v]$ or $\sigma_{1,2} = |\sigma| \exp(\pm \mathrm{i}\chi)$.

Substituting in (2.4), we have

$$u(t_n) = 2|\sigma|\cos(\chi + nz). \tag{2.5}$$

Now let us assume that the eigenvalues of D are ρ_1, ρ_2 and the corresponding eigenvectors are $[1, v_1]^{\mathrm{T}}$, $[1, v_2]^{\mathrm{T}}$, $v_i = A'/(\rho_i - B')$, $i = 1, 2$. The numerical solution of (1.5) is

$$u_n = c_1 \rho_1^n + c_2 \rho_2^n,$$

where

$$c_1 = -\frac{v_2 u_0 - h\dot{u}_0}{v_1 - v_2}, \qquad c_2 = \frac{v_1 u_0 - h\dot{u}_0}{v_1 - v_2}. \tag{2.6}$$

If ρ_1, ρ_2 are complex conjugate, then $c_{1,2} = |c| \exp(\pm \mathrm{i}w)$ and $\rho_{1,2} = |\rho| \exp(\pm \mathrm{i}p)$. By substituting in (2.6), we have

$$u_n = 2|c|\,|\rho|^n \cos(w + np). \tag{2.7}$$

Eqs. (2.5) and (2.7) lead us to the following definition.

Definition 2.1 (*Phase-lag*). Apply the RKN method (2.2) to (1.5). Then we define the *phase-lag* $\phi(z) = z - p$. If $\phi(z) = O(z^{q+1})$, then the RKN method is said to have phase-lag order q.

Additionally, the quantity $\alpha(z) = 1 - |\rho|$ is called *amplification error*.

Let us denote

$$R(z^2) = A(z^2) + B'(z^2) \quad \text{and} \quad Q(z^2) = A(z^2)B'(z^2) - A'(z^2)B(z^2). \tag{2.8}$$

From Definition 2.1 it follows that

$$\phi(z)=z-\arccos\left(\frac{R(z^2)}{2\sqrt{Q(z^2)}}\right), \quad |\rho|=\sqrt{Q(z^2)}. \tag{2.9}$$

If at a point z, $\alpha(z)=0$, then the RKN method has zero dissipation at this point. Thus we arrived at the following definition.

Definition 2.2 (*Interval of periodicity*). The *interval of periodicity* (or of *zero dissipation*) is the interval $[0, \gamma^2]$ on which $|\rho|=1$ ($\alpha(z)=0$) and $\rho_1 \neq \rho_2$.

For the interval of periodicity (PI) there is the following theorem.

Theorem 2.3. *If there is a nonempty interval of periodicity, then*

$$\alpha(z)=1-|\rho|=0 \quad \Leftrightarrow \quad R(z^2)=2.$$

Proof. It is obvious that (i) $\mathrm{Re}(\rho(z))=\frac{1}{2}R(z^2)$, (ii) $|\rho(z)| \leqslant 1$ and (iii) $\mathrm{Re}(\rho(z)) \leqslant |\rho(z)|$. Then if $\alpha(z)=0$, it follows from (iii) that $\frac{1}{2}R(z^2) \leqslant 1$, but on the other hand, it is obvious that $\frac{1}{2}R(z^2) \geqslant 1$ and thus $R(z^2)=2$.

If $R(z^2)=2$, it follows from (i)–(iii) that $1 \leqslant |\rho(z)| \leqslant 1$. Thus $\alpha(z)=0$ and the proof is complete. □

Now let us write $R(z^2)$ and $Q(z^2)$ in the form

$$R(z^2)=2-r_1z^2+r_2z^4-r_3z^6+\cdots+r_iz^{2i}=0, \qquad \text{for } i>m,$$
$$Q(z^2)=1-q_1z^2+q_2z^4-q_3z^6+\cdots+q_iz^{2i}=0, \qquad \text{for } i>m.$$

Van der Houwen and Sommeijer [16] derived, among others, the necessary conditions for the fourth-order accurate RKN method (2.2) to have phase-lag order eight in terms of r_i and q_i. These are

$$\begin{aligned} &r_1=1, \quad r_2=\tfrac{1}{12}, \quad q_1=q_2=9,\\ &r_3-q_3=\tfrac{1}{360}, \quad r_3+2r_4-2q_3-2q_4=\tfrac{29}{10080}. \end{aligned} \tag{2.10}$$

We will use these expressions to construct our eighth-order dispersive RKNF method.

3. The new method

In the following we shall derive a special four-stage fourth-order accurate eighth-order dispersive RKNF method, taking into account the dispersion relations (2.10).

Let us represent the general m-stage RKN method with the array

$$
\begin{array}{c|ccccc}
\alpha_1 & \gamma_{10} & & & & \\
\alpha_2 & \gamma_{20} & \gamma_{21} & & & \\
\vdots & \vdots & \vdots & & & \\
\alpha_{m-1} & \gamma_{m-1,0} & \gamma_{m-1,1} & \cdots & \gamma_{m-1,m-2} & \\
\hline
 & c_0 & c_1 & \cdots & c_{m-2} & c_{m-1} \\
 & \dot{c}_0 & \dot{c}_1 & \cdots & \dot{c}_{m-2} & \dot{c}_{m-1}
\end{array}
$$

The parameters above must satisfy the following relations in order to obtain fourth-order accuracy:

$$
\begin{aligned}
&c_0 + c_1 + c_2 + c_3 = \tfrac{1}{2}, && \dot{c}_0 + \dot{c}_1 + \dot{c}_2 + \dot{c}_3 = 1,\\
&c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \tfrac{1}{6}, && \dot{c}_1\alpha_1 + \dot{c}_2\alpha_2 + \dot{c}_3\alpha_3 = \tfrac{1}{2},\\
&c_1\alpha_1^2 + c_2\alpha_2^2 + c_3\alpha_3^2 = \tfrac{1}{12}, && \dot{c}_1\alpha_1^2 + \dot{c}_2\alpha_2^2 + \dot{c}_3\alpha_3^2 = \tfrac{1}{3},\\
&c_1\alpha_1^3 + c_2\alpha_2^3 + c_3\alpha_3^3 = \tfrac{1}{20}, && \dot{c}_1\alpha_1^3 + \dot{c}_2\alpha_2^3 + \dot{c}_3\alpha_3^3 = \tfrac{1}{4},\\
&c_2\gamma_{21}\alpha_1 + \tfrac{1}{6}c_3 = \tfrac{1}{120}, && \dot{c}_2\gamma_{21}\alpha_1 + \dot{c}_3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) = \tfrac{1}{24}.
\end{aligned}
\tag{3.1}
$$

First of all, we computed the polynomials A, B, A', B' and R and Q in terms of the RKN parameters leading us to the expressions shown in Appendix B.

We consider α_1 and α_2 as free parameters and put $\alpha_3 = 1$. Then we solved relations (3.1) and we obtained all the RKN parameters in terms of α_1 and α_2. This led us to the derivation of A, A', B, B', and therefore of $R(z^2)$ and $Q(z^2)$, in terms of α_1 and α_2.

In order to achieve phase-lag of order eight, relations (2.10) had to be satisfied simultaneously. The first row of relations (2.10) had been already satisfied because of the consistency conditions (3.1), thus the method already had fourth phase-lag order. The rest of relations (3.1) led us to the following systems of equations for α_1 and α_2:

$$
\begin{aligned}
&-10\alpha_2^3(87\alpha_1^3 - 154\alpha_1^2 + 88\alpha_1 - 16) + \alpha_2^2(420\alpha_1^4 + 740\alpha_1^3 - 2247\alpha_1^2 + 1480\alpha_1 - 288)\\
&\quad - \alpha_2(400\alpha_1^4 + 140\alpha_1^3 - 1146\alpha_1^2 + 849\alpha_1 - 174) + 2(40\alpha_1^4 + 15\alpha_1^3 - 117\alpha_1^2 + 87\alpha_1 - 18)\\
&\quad = 0,\\
&-210\alpha_2^4(203\alpha_1^5 - 688\alpha_1^4 + 887\alpha_1^3 - 538\alpha_1^2 + 152\alpha_1 - 16)\\
&\quad + 7\alpha_2^3(6090\alpha_1^6 + 10\,900\alpha_1^5 - 77\,915\alpha_1^4 + 116\,682\alpha_1^3 - 75\,838\alpha_1^2 + + 22\,560\alpha_1 - 2504)\\
&\quad - \alpha_2^2(127\,330\alpha_1^6 - 126\,685\alpha_1^5 - 443\,922\alpha_1^4 + 927\,218\alpha_1^3 - 671\,644\alpha_1^2 + 211\,940\alpha_1\\
&\qquad - 24\,552) + \alpha_2(90\,720\alpha_1^6 - 143\,685\alpha_1^5 - 138\,187\alpha_1^4 + 435\,091\alpha_1^3 - 343\,875\alpha_1^2\\
&\qquad\qquad + 113\,292\alpha_1 - 13\,524)\\
&-2(8260\alpha_1^6 - 12\,140\alpha_1^5 - 15\,535\alpha_1^4 + 43\,362\alpha_1^3 - 33\,729\alpha_1^2 + 11\,100\alpha_1 - 1332) = 0.
\end{aligned}
$$

By solving numerically, using the Newton–Raphson method, from these equations the parameters α_1 and α_2 were obtained and therefore all the RKNF parameters were determined.

The method constructed in this way is represented by the array in (A.1) in Appendix A.
Additionally by applying Theorem 2.3 to our method, we determined its interval of periodicity $[0, \gamma^2] = [0, 9.114475]$.

4. Numerical examples

In this section we will apply our method to four problems. For purposes of illustration, we will compare our results with those derived by using two methods of [16], which are represented by the arrays in (A.2) and (A.3), together with the order of accuracy p, the order of phase-lag q and the periodicity interval PI. We note that equations of the form of the examples presented below can be found very frequently in astrophysics, theoretical physics, agriculture, biochemistry, etc. [9].

Problem 4.1.

$$\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} = -v^2u(t) + (v^2 - 1)\sin(t), \quad u(0) = 1, \ u'(0) = v + 1, \ t \geqslant 0,$$

where $v \gg 1$. The exact solution is $u(t) = \cos(vt) + \sin(vt) + \sin(t)$. Numerical results are given for $v = 10$.

Problem 4.2.

$$\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} = -u(t) + \epsilon \exp(\mathrm{i}t), \quad u(t) \in \mathbb{C}, \ u(0) = 1, \ u'(0) = \left(1 - \tfrac{1}{2}\epsilon\right)\mathrm{i}.$$

The exact solution of the above equation is

$$u(t) = \cos(t) + \tfrac{1}{2}\epsilon t \sin(t) + \mathrm{i}\left[\sin(t) - \tfrac{1}{2}\epsilon t \cos(t)\right].$$

Numerical results are given for $\epsilon = 0.001$.

Problem 4.3 (*Nonlinear equation*).

$$\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} = -u(t) - (u(t))^3 + B\cos(vt),$$

where $B = 0.002$ and $v = 1.01$. The exact solution computed by the Galerkin method with a precision 10^{-12} of the coefficients is given by

$$u(t) = A_1\cos(vt) + A_3\cos(3vt) + A_5\cos(5vt) + A_7\cos(7vt) + A_9\cos(9vt),$$

where

$$A_1 = 0.200\,179\,477\,536, \qquad A_3 = 0.000\,246\,946\,143, \qquad A_5 = 0.000\,000\,304\,014,$$
$$A_7 = 0.000\,000\,000\,374, \qquad A_9 = 0.000\,000\,000\,000.$$

Table 1
sd-values for the maximum absolute error for Problems 4.1–4.4

		Our method (A.1) $p=4$; $q=8$			Van der Houwen's method (A.2) $p=2$; $q=8$			Van der Houwen's method (A.3) $p=4$; $q=10$		
		$T=100$	$T=1000$	$T=4000$	$T=100$	$T=1000$	$T=4000$	$T=100$	$T=1000$	$T=4000$
Problem 4.1	$h=0.025$	4.3	3.3	2.7	1.1	0.1	–	2.3	2.3	2.3
	$h=0.050$	2.8	1.8	1.8	0.0	–	–	1.7	1.7	1.7
Problem 4.2	$h=0.25$	5.4	4.4	3.8	2.3	2.3	2.3	2.3	1.3	0.7
	$h=0.50$	3.9	2.9	2.2	1.7	1.7	1.5	1.1	0.1	–
$(\epsilon=0.001)$										
Problem 4.3	$h=0.25$	5.7	5.5	5.5	2.9	2.8	2.8	2.9	2.7	2.7
	$h=0.50$	4.3	4.1	4.1	2.3	2.2	2.2	1.6	1.4	1.4
Problem 4.4	$h=0.025$	2.1	2.0	2.0	0.6	0.5	0.5	0.2	0.1	0.1
	$h=0.050$	1.5	1.4	1.4	–	–	–	–	–	–

Empty areas denote negative sd-values.

Problem 4.4 (*Wave equation*).

$$\frac{\partial^2 u}{\partial t^2}=\frac{b^2}{4\pi^2}\frac{\partial^2 u}{\partial x^2},\quad 0\leqslant x\leqslant b,\ t\geqslant 0,$$

$$\frac{\partial u}{\partial x}(t,0)=0=\frac{\partial u}{\partial x}(t,b),$$

$$u(0,x)=\cos(x),\quad \frac{\partial u}{\partial t}(0,x)=0.$$

We used second-order symmetric differences to convert this problem into a system of ODEs. Numerical results are given for $\Delta x=1$, $b=5$.

The results for the four problems above are listed in Table 1. One measure of the accuracy of a method is to compute the sd-value for the maximum error, that is, the number of correct decimal digits

$$\mathrm{sd}(T)=-\log_{10}(\max|u(t_n)-u_n|),\quad \text{where } t_n=1+nh,\ n=1,2,\ldots,\frac{T-1}{h}.$$

Table 1 shows the sd-values for methods (A.1)–(A.3) of Appendix A for two step values, $h=0.025$ and $h=0.050$ for Problems 4.1 and 4.4, and $h=0.25$ and $h=0.50$ for Problems 4.2 and 4.3.

5. Conclusion

Numerical results show that the new method is much more powerful for integrating second-order equations possessing an oscillatory solution than conventional RKN methods.

Furthermore, the new method is simple and, without being computationally expensive, is more accurate than existing methods dealing with the problems considered in the present paper.

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Appendix A

$$
\begin{array}{c|cccc}
0.254\,752\,951\,59 & 0.032\,449\,532\,99 & & & \\
0.505\,403\,169\,62 & 0.032\,922\,844\,93 & 0.094\,793\,336\,59 & & \\
1 & 0.190\,145\,049\,13 & 0 & 0.309\,854\,951\,35 & \\
\hline
 & 0.100\,227\,915\,53 & 0.184\,580\,085\,59 & 0.193\,182\,906\,96 & 0.022\,009\,091\,91 \\
 & 0.163\,236\,038\,47 & 0.018\,923\,885\,31 & 0.652\,371\,780\,35 & 0.165\,468\,328\,71
\end{array}
\tag{A.1}
$$

$p = 4$, $q = 8$, PI $= [0, 9.114475]$.

$$
\begin{array}{c|cccc}
0.055\,159\,431\,7 & 0.001\,521\,281\,5 & & & \\
0.668\,370\,144\,6 & -1.173\,201\,611\,6 & 1.396\,560\,936\,7 & & \\
0.363\,210\,962\,8 & 1.588\,740\,385\,5 & -1.726\,328\,914\,5 & 0.203\,549\,630\,8 & \\
\hline
 & 0.404\,644\,025\,0 & -0.346\,469\,679\,9 & 0.082\,913\,499\,9 & 0.358\,912\,155\,0 \\
 & -1.806\,738\,925\,1 & 2.641\,099\,086\,4 & 0.963\,943\,697\,1 & -0.798\,303\,858\,4
\end{array}
\tag{A.2}
$$

$p = 4$, $q = 10$, PI $= [0, (3.59)^2]$.

$$
\begin{array}{c|ccccc}
\frac{1}{2} & 0 & & & & \\
\frac{1}{2} & 0 & \frac{1}{56} & & & \\
\frac{1}{2} & 0 & 0 & \frac{1}{30} & & \\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{12} & \\
\hline
 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
 & 0 & 0 & 0 & 0 & 1
\end{array}
\tag{A.3}
$$

$p = 2$, $q = 8$, PI $= [0, (4.63)^2]$.

Appendix B

$$
\begin{aligned}
A(z^2) = {} & c_3\gamma_{10}\gamma_{21}\gamma_{32}z^8 - \left(c_2\gamma_{10}\gamma_{21} + c_3\left(\gamma_{10}\gamma_{31} + \gamma_{32}(\gamma_{20} + \gamma_{21})\right)\right)z^6 \\
& + \left(c_1\gamma_{10} + c_2(\gamma_{20} - \gamma_{21}) + c_3(\gamma_{30} + \gamma_{31} + \gamma_{32})\right)z^4 - (c_0 + c_1 + c_2 + c_3)z^2 + 1,
\end{aligned}
$$

$$
\begin{aligned}
B(z^2) = {} & -\alpha_1 c_3 \gamma_{21} \gamma_{32} z^6 + (\alpha_1(c_2\gamma_{21} + c_3\gamma_{31}) + \alpha_2 c_3 \gamma_{32}) z^4 - (\alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3) z^2 + 1, \\
A'(z^2) = {} & \dot{c}_3 \gamma_{10}\gamma_{21}\gamma_{32} z^8 - (\dot{c}_2\gamma_{10}\gamma_{21} + \dot{c}_3(\gamma_{10}\gamma_{31} + \gamma_{32}(\gamma_{20} + \gamma_{21}))) z^6 \\
& + (\dot{c}_1\gamma_{10} + \dot{c}_2(\gamma_{20} + \gamma_{21}) + \dot{c}_3(\gamma_{30} + \gamma_{31} + \gamma_{32})) z^4 - (\dot{c}_0 + \dot{c}_1 + \dot{c}_2 + \dot{c}_3) z^2, \\
B'(z^2) = {} & -\alpha_1 \dot{c}_3 \gamma_{21}\gamma_{32} z^6 + (\alpha_1(\dot{c}_2\gamma_{21} + \dot{c}_3\gamma_{31}) + \alpha_2 \dot{c}_3 \gamma_{32}) z^4 - (\alpha_1 \dot{c}_1 + \alpha_2 \dot{c}_2 + \alpha_3 \dot{c}_3) z^2 + 1, \\
R(z^2) = {} & c_3 \gamma_{10}\gamma_{21}\gamma_{32} z^8 - (\alpha_1 \dot{c}_3 \gamma_{21}\gamma_{32} + c_2\gamma_{10}\gamma_{21} + c_3(\gamma_{10}\gamma_{31} + \gamma_{32}(\gamma_{20} + \gamma_{21}))) z^6 \\
& + (\alpha_1(\dot{c}_2\gamma_{21} + \dot{c}_3\gamma_{31}) + \alpha_2\dot{c}_3\gamma_{32} + c_1\gamma_{10} + c_2(\gamma_{20} + \gamma_{21}) + c_3(\gamma_{30} + \gamma_{31} + \gamma_{32})) z^4 \\
& - (\alpha_1\dot{c}_1 + \alpha_2\dot{c}_2 + \alpha_3\dot{c}_3 + c_0 + c_1 + c_2 + c_3) z^2 + 2, \\
Q(z^2) = {} & (\alpha_1(c_0\dot{c}_3\gamma_{21}\gamma_{32} - c_1\dot{c}_3\gamma_{32}\gamma_{20} + c_2\dot{c}_3(\gamma_{20}\gamma_{31} - \gamma_{21}\gamma_{30}) - c_3(\dot{c}_0\gamma_{21}\gamma_{32} - \dot{c}_1\gamma_{20}\gamma_{32} \\
& + \dot{c}_2(\gamma_{20}\gamma_{31} - \gamma_{30}\gamma_{21}))) + \alpha_2\gamma_{10}(c_1\dot{c}_3\gamma_{32} - c_2\dot{c}_3\gamma_{31} + c_3(\dot{c}_2\gamma_{31} - \dot{c}_1\gamma_{32})) \\
& + \alpha_3\gamma_{10}\gamma_{21}(c_2\dot{c}_3 - c_3\dot{c}_2) + c_3\gamma_{10}\gamma_{21}\gamma_{32} - \dot{c}_3\gamma_{10}\gamma_{21}\gamma_{32}) z^8 - (\alpha_1(c_0(\dot{c}_2\gamma_{21} + \dot{c}_3\gamma_{31}) \\
& - c_1(\dot{c}_2\gamma_{20} + \dot{c}_3(\gamma_{30} + \gamma_{32})) - c_2(\dot{c}_0\gamma_{21} - \dot{c}_1\gamma_{20} + \dot{c}_3(\gamma_{21} - \gamma_{31})) \\
& - c_3(\dot{c}_0\gamma_{31} - \dot{c}_1(\gamma_{30} + \gamma_{32}) \\
& + \dot{c}_2(\gamma_{31} - \gamma_{21}) + \dot{c}_3\gamma_{21}\gamma_{32}) + \alpha_2 c_0\dot{c}_3\gamma_{32} + c_1(\dot{c}_2\gamma_{10} + \dot{c}_3\gamma_{32})) \\
& - c_2(\dot{c}_1\gamma_{10} + \dot{c}_3(\gamma_{31} + \gamma_{30}) - c_3(\dot{c}_0\gamma_{32} + \dot{c}_1\gamma_{32}) - \dot{c}_2(\gamma_{31} + \gamma_{30})) \\
& + \alpha_3(c_1\dot{c}_3\gamma_{10} + \dot{c}_3 c_2(\gamma_{20} + \gamma_{21}) - c_3(\dot{c}_1\gamma_{10} + \dot{c}_2(\gamma_{21} + \gamma_{20}))) + c_2\gamma_{20}\gamma_{21} \\
& + c_3(\gamma_{10}\gamma_{31} + \gamma_{32}(\gamma_{20} + \gamma_{21})) - \dot{c}_2\gamma_{20}\gamma_{21} - \dot{c}_3(\gamma_{10}\gamma_{31} + \gamma_{32}(\gamma_{20} + \gamma_{21}))) z^6 \\
& + (\alpha_1(c_0\dot{c}_1 - c_1(\dot{c}_0 + \dot{c}_2 + \dot{c}_3) + c_2\dot{c}_1 + c_3\dot{c}_1 + \dot{c}_2\gamma_{21} + \dot{c}_3\gamma_{31}) \\
& + \alpha_2(c_0\dot{c}_2 + c_1\dot{c}_2 - c_2(\dot{c}_0 + \dot{c}_1 + \dot{c}_3) + c_3\dot{c}_2 + \dot{c}_3\gamma_{32}) + \alpha_3(c_0\dot{c}_3 + c_1\dot{c}_3 + c_2\dot{c}_3 \\
& - c_3(\dot{c}_0 + \dot{c}_1 + \dot{c}_2)) + c_1\gamma_{10} + c_2(\gamma_{20} + \gamma_{21}) + c_3(\gamma_{30} + \gamma_{31} + \gamma_{32}) \\
& - \dot{c}_1\gamma_{10} - \dot{c}_2(\gamma_{20} + \gamma_{21}) - \dot{c}_3(\gamma_{30} + \gamma_{31} + \gamma_{32})) z^4 \\
& - (\alpha_1\dot{c}_1 + \alpha_2\dot{c}_2 + \alpha_3\dot{c}_3 + c_0 + c_1 + c_2 + c_3 - \dot{c}_0 - \dot{c}_1 - \dot{c}_2 - \dot{c}_3) + 1.
\end{aligned}
$$

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